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# Parametric Bivariate Regression Analysis Based on Censored Samples: A Weibull Model

**David D. Hanagal**

**Abstract:** In this paper, we propose a new bivariate Weibull distribution and derive estimation and test procedures based on censored samples with common covariates. The study of bivariate Weibull was motivated by interesting biometrical applications. The maximum likelihood estimators for the model parameters are obtained together with a test of significance.

*Key words:* Bivariate Weibull model, Parametric regression, Survival times.

## 1 Introduction

Hanagal [3] proposed a multivariate Weibull distribution which includes the multivariate exponential of Marshall-Olkin [4]. Thus, our bivariate Weibull (BVW) model represents a super model of the BVE of Marshall-Olkin [4]. Some biometrical applications have motivated to study a BVW regression model in connection with paired organs like kidneys, eyes, ears or others. These pairs of an individual (e.g. patient) is considered as a two component system. The components are assumed to be dependent, as it is quite common that for instance simultaneous failure of organs may occur.

The life time of an individual is assumed to be independent of the life times of the paired organs. Because death of an individual will censor both life times of organs, it is possible to use univariate censoring as given by Hanagal [1, 2].

In the above mentioned biometrical applications, the covariates were age, sex, smoking or alcoholic habits of the patient, diabetic or non-diabetic conditions or some specific diseases of the patient etc. In such situations, it is not realistic to treat the life times of the paired organs as identically distributed BVW, as it depends heavily of the common covariates. These covariates are the characteristics or properties of a patient and are not different for each organ of an individual.

In Section 2, the BVW distribution is introduced; Section 3 is devoted to the maximum likelihood estimator of the parameters, and in Section 4, a significance test for the regression parameters is briefly outlined.

## 2 Bivariate Weibull Distribution (BVW)

Hanagal [3] proposed a $(k+2)$ parameter family of $k$-variate Weibull distributions. The BVW is obtained by specifying $k = 2$; it has the following probability density function:

$$f(t_1,t_2) = \begin{cases} \lambda_1(\lambda_2+\lambda_3)\sigma^2(t_1t_2)^{\sigma-1}e^{-\lambda_1 t_1^\sigma-(\lambda_2+\lambda_3)t_2^\sigma} & \text{for} \quad 0 < t_1 < t_2 < \infty \\ \lambda_2(\lambda_1+\lambda_3)\sigma^2(t_1t_2)^{\sigma-1}e^{-\lambda_2 t_2^\sigma-(\lambda_1+\lambda_3)t_1^\sigma} & \text{for} \quad 0 < t_2 < t_1 < \infty \\ \lambda_3\sigma t^{\sigma-1}e^{-(\lambda_1+\lambda_2+\lambda_3)t^\sigma} & \text{for} \quad 0 < t_1 = t_2 = t < \infty \end{cases} \tag{1}$$

where $\lambda_1, \lambda_2, \lambda_3 > 0$.

The BVW distribution (1) has remarkable properties. Both marginal distributions of BVW are univariate Weibull. More specifically speaking, we can state that

Let $(T_1, T_2) \sim \text{BVW}(\lambda_1, \lambda_2, \lambda_3; \sigma)$ then the following implications hold:

- $T_1 \sim \text{Weibull}((\lambda_1+\lambda_3), \sigma)$ (2)

- $T_2 \sim \text{Weibull}((\lambda_2+\lambda_3), \sigma)$ (3)

- $T_3 = \min(T_1, T_2) \sim \text{Weibull}((\lambda_1+\lambda_2+\lambda_3), \sigma)$ (4)

where $(\lambda_1+\lambda_3)$, $(\lambda_2+\lambda_3)$ and $(\lambda_1+\lambda_2+\lambda_3)$ are the scale parameters of $T_1$, $T_2$ and $T_3$, respectively and $\sigma$ is the common shape parameter. The parameter $\lambda_3$ quantifies the dependence between the two variables $(T_1, T_2)$ and $\lambda_3 = 0$ implies $T_1$ and $T_2$ are independent.

The BVW distribution (1) is not absolutely continuous with respect to Lebesque measure in $R^2$. It has a singularity on the diagonal $t_1 = t_2$, which represents simultaneous failure of $T_1$ and $T_2$

The probability of simultaneous failures is easily obtained:

$$P(T_1 = T_2) = \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3} \tag{5}$$

which also gives the correlation between $T_1$ and $T_2$.

The diagonal $t_1 = t_2$ separates the plane in two regions. The probability for these are the following:

$$P(T_1 < T_2) = \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \tag{6}$$

$$P(T_1 > T_2) = \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} \tag{7}$$

From (1) the BVE of Marshall-Olkin [4] is obtainde for $\sigma = 1$. Taking the logarithm of BVE variables i.e., $Y_1$=log$T_1$ and $Y_2$=log$T_2$ leads to a bivariate extreme value(BVEV) distribution with the following probability density function:

$$f(y_1, y_2) = \begin{cases} e^{y_1+\log\lambda_1+y_2+\log(\lambda_2+\lambda_3)-e^{y_1+\log\lambda_1}-e^{y_2+\log(\lambda_2+\lambda_3)}} & \text{for} \quad -\infty < y_1 < y_2 < \infty \\ e^{y_2+\log\lambda_2+y_1+\log(\lambda_1+\lambda_3)-e^{y_2+\log\lambda_2}-e^{y_1+\log(\lambda_1+\lambda_3)}} & \text{for} \quad -\infty < y_2 < y_1 < \infty \\ e^{y+\log\lambda_3-e^{y+\log(\lambda_1+\lambda_2+\lambda_3)}} & \text{for} \quad -\infty < y_1 = y_2 = y < \infty \end{cases} \tag{8}$$

From (1) and (8) it can be seen that $(\lambda_1, \lambda_2, \lambda_3)$ are not proper scale parameters for BVE of Marshall-Olkin(1967). In the univariate case, the location parameter of univariate extreme value distribution is the scale parameter of the exponential distribution. Thus, BVE of Marshall-Olkin(1967) and corresponding BVEV do not belong to the location-scale family.

Assuming that the variables of interest $(Y_1, Y_2)$ depend on some covariates given by $(\mathbf{X}_1, \mathbf{X}_2)$ leads to regarding the following regression model for the two component system,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1^{'}\mathbf{X}_1 \\ \mathbf{b}_2^{'}\mathbf{X}_2 \end{pmatrix} + \frac{1}{\sigma}\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \tag{9}$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are $m$-dimensional vectors of covariates, $\mathbf{b}_1$ and $\mathbf{b}_2$ are $m$-dimensional vectors of regression coefficients and $(U_1, U_2)$ follows a BVEV distribution as given by (8). Knowing that the BVEV distribution does not belong to the location-scale family, there is no intercept term included in the regression model.

Taking $Y_1 = \log T_1$ and $Y_2 = \log T_2$, we obtain

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} e^{\mathbf{b}_1^{'}\mathbf{X}_1} V_1^{1/\sigma} \\ e^{\mathbf{b}_2^{'}\mathbf{X}_2} V_2^{1/\sigma} \end{pmatrix} \tag{10}$$

where $(V_1 = e^{U_1},\ V_2 = e^{U_2})$ is BVE of Marshall-Olkin(1967). Alternately we may write

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} e^{-\sigma\mathbf{b}_1^{'}\mathbf{X}_1} T_1^{\sigma} \\ e^{-\sigma\mathbf{b}_2^{'}\mathbf{X}_2} T_2^{\sigma} \end{pmatrix} \tag{11}$$

Making the additional assumption that both components have common covariates and regression parameters, i.e., $\mathbf{X}_1 = \mathbf{X}_2$ and $\mathbf{b}_1^{'} = \mathbf{b}_2^{'}$, then we have

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} T_1^{\sigma} \\ T_2^{\sigma} \end{pmatrix} e^{-\sigma\mathbf{b}^{'}\mathbf{X}} \tag{12}$$

## 3 ML-Estimation of the Parameters

The univariate random censoring scheme given by Hanagal [1, 2] is used for estimating the bivariate life time distribution, which takes into account that individuals do not enter at the same time the study and a withdrawal of an individual will censor both life times of the components which in the sequel will be called implants, because the model was developed and applied in the framework of teeth implants for upper and lower jaws.

Suppose that there are $n$ independent pairs of implants under study, where the $i$th pair of implants have life times $(\tilde{T}_{1i}, \tilde{T}_{2i})$ and a censoring time $(Z_i)$.

Let the censored random life of the $i$th pair be denoted by $(T_{1i}, T_{2i})$. Then $(T_{1i}, T_{2i})$ is defined as follows:

$$(T_{1i}, T_{2i}) = \begin{cases} (\tilde{T}_{1i}, \tilde{T}_{2i}) & \text{if} \quad \max(\tilde{T}_{1i}, \tilde{T}_{2i}) < Z_i \\ (\tilde{T}_{1i}, Z_i) & \text{if} \quad \tilde{T}_{1i} < Z_i < \tilde{T}_{2i} \\ (Z_i, \tilde{T}_{2i}) & \text{if} \quad \tilde{T}_{2i} < Z_i < \tilde{T}_{1i} \\ (Z_i, Z_i) & \text{if} \quad Z_i < \min(\tilde{T}_{1i}, \tilde{T}_{2i}) \end{cases} \tag{13}$$

There are six different types of events which might occur with respect to $(T_{1i}, T_{2i})$, $i = 1, \ldots, n$. These are the following:

1. Type 1: $\tilde{T}_{1i} < \tilde{T}_{2i} < Z_i$
2. Type 2: $\tilde{T}_{2i} < \tilde{T}_{1i} < Z_i$
3. Type 3: $\tilde{T}_{1i} = \tilde{T}_{2i} < Z_i$
4. Type 4: $\tilde{T}_{1i} < Z_i < \tilde{T}_{2i}$
5. Type 5: $\tilde{T}_{2i} < Z_I < \tilde{T}_{1i}$
6. Type 6: $Z_i < \min(\tilde{T}_{1i}, \tilde{T}_{2i})$

Below, the probability density functions for Type 1 to 5 and the survival function for Type 6 of $(T_{1i}, T_{2i})$ are stated.

$$f_1(t_1, t_2) = \sigma^2 \lambda_1 (\lambda_2 + \lambda_3)(t_1 t_2)^{\sigma-1} e^{-2\sigma \mathbf{X}'\mathbf{b} - [\lambda_1 t_1^\sigma + (\lambda_2 + \lambda_3) t_2^\sigma] e^{-\sigma \mathbf{X}'\mathbf{b}}} \tag{14}$$

$$f_2(t_1, t_2) = \sigma^2 \lambda_2 (\lambda_1 + \lambda_3)(t_1 t_2)^{\sigma-1} e^{-2\sigma \mathbf{X}'\mathbf{b} - [\lambda_2 t_2^\sigma + (\lambda_1 + \lambda_3) t_1^\sigma] e^{-\sigma \mathbf{X}'\mathbf{b}}} \tag{15}$$

$$f_3(t, t) = \sigma \lambda_3 t^{\sigma-1} e^{-\sigma \mathbf{X}'\mathbf{b} - (\lambda_1 + \lambda_2 + \lambda_3) t^\sigma e^{-\sigma \mathbf{X}'\mathbf{b}}} \tag{16}$$

$$f_4(t_1, t_2) = \lim_{\delta t \to 0} \frac{P[t_1 < \tilde{T}_1 < t_1 + \delta t \mid \tilde{T}_2 > t_2] P[\tilde{T}_2 > t_2]}{\delta t} \tag{17}$$

$$= \sigma \lambda_1 t_1^{\sigma-1} e^{-\sigma \mathbf{X}'\mathbf{b} - [\lambda_1 t_1^\sigma + (\lambda_2 + \lambda_3) t_2^\sigma] e^{-\sigma \mathbf{X}'\mathbf{b}}} \tag{18}$$

$$
\begin{aligned}
f_5(t_1,t_2) &= \lim_{\delta t\to\ 0}\frac{P[t_2<\tilde{T}_2<t_2+\delta t\mid \tilde{T}_1>t_1]P[\tilde{T}_1>t_1]}{\delta t_i} && (19)\\
&= \sigma\lambda_2 t_2^{\sigma-1}e^{-\sigma\mathbf{X}'\mathbf{b}-[\lambda_2 t_2^\sigma+(\lambda_1+\lambda_3)t_1^\sigma]e^{-\sigma\mathbf{X}'\mathbf{b}}} && (20)
\end{aligned}
$$

$$
\overline{F}(t,t) = P[T_1>t,T_2>t]=e^{-(\lambda_1+\lambda_2+\lambda_3)t^\sigma e^{-\sigma\mathbf{X}'\mathbf{b}}} \quad (21)
$$

where

$$
\begin{aligned}
\mathbf{X} &= (\tilde{T}_1,...,\tilde{T}_m)' && (22)\\
\mathbf{b} &= (b_1,...,b_m)' && (23)
\end{aligned}
$$

Let $n_1$, $n_2$, $n_3$, $n_4$, $n_5$ and $n_6$ be the numbers of observations representing the different types of events with $n=n_1+\ldots+n_6$. Then the likelihood function $L$ for a sample $\Big((t_{11,t_21}),\ldots,t_1n,t_{2n})\Big)$ is given as follows:

$$
L=\left(\prod_{i=1}^{n_1}f_1\right)\left(\prod_{i=1}^{n_2}f_2\right)\left(\prod_{i=1}^{n_3}f_3\right)\left(\prod_{i=1}^{n_4}f_1\right)\left(\prod_{i=1}^{n_5}f_2\right)\left(\prod_{i=1}^{n_6}\overline{F}\right) \quad (24)
$$

Discarding factors which do not contain any of the parameters and taking the logarithm yields:

$$
\begin{aligned}
\log L &= (2n_1+2n_2+n_3+n_4+n_5)\log\sigma+(n_1+n_4)\log\lambda_1+(n_2+n_5)\log\lambda_2+n_3\log\lambda_3+\\
&\quad n_1\log(\lambda_2+\lambda_3)+n_2\log(\lambda_1+\lambda_3)+(\sigma-1)\sum_{i\in A}\log t_{1i}+(\sigma-1)\sum_{i\in B}\log t_{2i}-\\
&\quad \sigma\sum_{i\in C}\mathbf{X}_i'\mathbf{b}-\sigma\sum_{i\in D}\mathbf{X}_i'\mathbf{b}-\sum_{i=1}^{n}(\lambda_1 t_{1i}^\sigma+\lambda_2 t_{2i}^\sigma+\lambda_3 t_{(2)i}^\sigma)e^{-\sigma\mathbf{X}_i'\mathbf{b}} && (25)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \{i\mid t_{1i}<z_i\} && (26)\\
B &= \{i\mid t_{2i}<z_i\} && (27)\\
C &= \{i\mid \max(t_{1i},t_{2i})<z_i, t_{1i}\neq t_{2i}\} && (28)\\
D &= \{i\mid t_{1i}<z_i \text{ or } t_{2i}<z_i\} && (29)\\
&= C\cup\{i\mid t_{1i}=t_{2i}<z_i\}\cup\{i|t_{(1)i}<z_i<t_{(2)i}\} && (30)
\end{aligned}
$$

with

$$
\begin{aligned}
t_{(1)i} &= \min(t_{1i},t_{2i}) && (31)\\
t_{(2)i} &= \max(t_{1i},t_{2i}) && (32)
\end{aligned}
$$

The likelihood equations are given by

$$\frac{\partial \log L}{\partial \lambda_1} = \frac{(n_1+n_4)}{\lambda_1} + \frac{n_2}{\lambda_1+\lambda_3} - \sum_{i=1}^{n} t_{1i}^{\sigma} e^{-\sigma \mathbf{X}_i' \mathbf{b}} = 0 \quad (33)$$

$$\frac{\partial \log L}{\partial \lambda_2} = \frac{(n_2+n_5)}{\lambda_2} + \frac{n_1}{\lambda_2+\lambda_3} - \sum_{i=1}^{n} t_{2i}^{\sigma} e^{-\sigma \mathbf{X}_i' \mathbf{b}} = 0 \quad (34)$$

$$\frac{\partial \log L}{\partial \lambda_3} = \frac{n_3}{\lambda_3} + \frac{n_1}{\lambda_2+\lambda_3} + \frac{n_2}{\lambda_1+\lambda_3} - \sum_{i=1}^{n} t_{(2)i}^{\sigma} e^{-\sigma \mathbf{X}_i' \mathbf{b}} = 0 \quad (35)$$

$$\begin{aligned}\frac{\partial \log L}{\partial \sigma} = {} & \frac{(2n_1+2n_2+n_3+n_4+n_5)}{\sigma} + \\ & \sum_{i\in A} \log t_{1i} + \sum_{i\in B} \log t_{2i} - \sum_{i\in C} \mathbf{X}_i' \mathbf{b} - \sum_{i\in D} \mathbf{X}_i' \mathbf{b} - \\ & \lambda_1 \sum_{i=1}^{n} t_{1i}^{\sigma} e^{-\sigma \mathbf{X}_i' \mathbf{b}} [\log t_{1i} - \mathbf{X}_i' \mathbf{b}] - \lambda_2 \sum_{i=1}^{n} t_{2i}^{\sigma} e^{-\sigma \mathbf{X}_i' \mathbf{b}} [\log t_{2i} - \mathbf{X}_i' \mathbf{b}] - \\ & \lambda_3 \sum_{i=1}^{n} t_{(2)i}^{\sigma} e^{-\sigma \mathbf{X}_i' \mathbf{b}} [\log t_{(2)i} - \mathbf{X}_i' \mathbf{b}] = 0 \end{aligned} \quad (36)$$

$$\frac{\partial \log L}{\partial b_j} = -\sigma \sum_{i\in C} \mathbf{X}_{ji} - \sigma \sum_{i\in D} \mathbf{X}_{ji} + \sigma \sum_{i=1}^{n} (\lambda_1 t_{1i}^{\sigma} + \lambda_2 t_{2i}^{\sigma} + \lambda_3 t_{(2)i}^{\sigma}) X_{ji} e^{-\sigma \mathbf{X}_i' \mathbf{b}} = 0 \quad (37)$$

for $j = 1, ..., m$.

The likelihood equations may be solved by a Newton-Raphson procedure, where the second order partial derivatives of the log-likelihood function are given by:

$$\frac{\partial^2 \log L}{\partial \lambda_1^2} = -\frac{(n_1+n_4)}{\lambda_1^2} - \frac{n_2}{(\lambda_1+\lambda_3)^2} \quad (38)$$

$$\frac{\partial^2 \log L}{\partial \lambda_2^2} = -\frac{(n_2+n_5)}{\lambda_2^2} - \frac{n_1}{(\lambda_2+\lambda_3)^2} \quad (39)$$

$$\frac{\partial^2 \log L}{\partial \lambda_3^2} = -\frac{n_3}{\lambda_3^2} - \frac{n_1}{(\lambda_2+\lambda_3)^2} - \frac{n_2}{(\lambda_1+\lambda_3)^2} \quad (40)$$

$$\frac{\partial^2 \log L}{\partial \lambda_1 \partial \lambda_2} = 0 \quad (41)$$

$$\frac{\partial^2 \log L}{\partial \lambda_1 \partial \lambda_3} = -\frac{n_2}{(\lambda_1+\lambda_3)^2} \quad (42)$$

$$\frac{\partial^2 \log L}{\partial \lambda_2 \partial \lambda_3} = -\frac{n_1}{(\lambda_2+\lambda_3)^2} \quad (43)$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial \sigma^2} = {} & -\frac{(2n_1+2n_2+n_3+n_4+n_5)}{\sigma^2} - \lambda_1 \sum_{i=1}^{n} t_{1i}^{\sigma} e^{-\sigma \mathbf{X}_i' \mathbf{b}} (\log t_{1i} - \mathbf{X}_i' \mathbf{b})^2 - \\ & \lambda_2 \sum_{i=1}^{n} t_{2i}^{\sigma} e^{-\sigma \mathbf{X}_i' \mathbf{b}} (\log t_{2i} - \mathbf{X}_i' \mathbf{b})^2 - \lambda_3 \sum_{i=1}^{n} t_{(2)i}^{\sigma} e^{-\sigma \mathbf{X}_i' \mathbf{b}} (\log t_{(2)i} - \mathbf{X}_i' \mathbf{b})^2 \end{aligned} \quad (44)$$

$$\frac{\partial^2 \log L}{\partial\sigma\partial\lambda_1} = -\sum_{i=1}^{n} t_{1i}^{\sigma} e^{-\sigma \mathbf{X}_i^{'}\mathbf{b}} (\log t_{1i} - \mathbf{X}_i^{'}\mathbf{b}) \tag{45}$$

$$\frac{\partial^2 \log L}{\partial\sigma\partial\lambda_2} = -\sum_{i=1}^{n} t_{2i}^{\sigma} e^{-\sigma \mathbf{X}_i^{'}\mathbf{b}} (\log t_{2i} - \mathbf{X}_i^{'}\mathbf{b}) \tag{46}$$

$$\frac{\partial^2 \log L}{\partial\sigma\partial\lambda_3} = -\sum_{i=1}^{n} t_{(2)i}^{\sigma} e^{-\sigma \mathbf{X}_i^{'}\mathbf{b}} (\log t_{(2)i} - \mathbf{X}_i^{'}\mathbf{b}) \tag{47}$$

$$\frac{\partial^2 \log L}{\partial b_j\partial b_k} = -\sigma^2 \sum_{i=1}^{n} (\lambda_1 t_{1i}^{\sigma} + \lambda_2 t_{2i}^{\sigma} + \lambda_3 t_{(2)i}^{\sigma}) X_{ji} X_{ki} e^{-\sigma \mathbf{X}_i^{'}\mathbf{b}} \qquad j,k = 1,...,m \tag{48}$$

$$\frac{\partial^2 \log L}{\partial b_j\partial\lambda_1} = \sigma \sum_{i=1}^{n} t_{1i}^{\sigma} X_{ji} e^{-\sigma \mathbf{X}_i^{'}\mathbf{b}} \qquad j = 1,...,m \tag{49}$$

$$\frac{\partial^2 \log L}{\partial b_j\partial\lambda_2} = \sigma \sum_{i=1}^{n} t_{2i}^{\sigma} X_{ji} e^{-\sigma \mathbf{X}_i^{'}\mathbf{b}} \qquad j = 1,...,m \tag{50}$$

$$\frac{\partial^2 logL}{\partial b_j\partial\lambda_3} = \sigma \sum_{i=1}^{n} t_{(2)i}^{\sigma} X_{ji} e^{-\sigma \mathbf{X}_i^{'}\mathbf{b}} \qquad j = 1,...,m \tag{51}$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial b_j\partial\sigma} = & -\sum_{i\in C} X_{ji} - \sum_{i\in D} X_{ji} + \sum_{i=1}^{n} (\lambda_1 t_{1i}^{\sigma} + \lambda_2 t_{2i}^{\sigma} + \lambda_3 t_{(2)i}^{\sigma}) X_{ji} e^{-\sigma \mathbf{X}_i^{'}\mathbf{b}} + \\ & \sigma \sum_{i=1}^{n} X_{ji} e^{-\sigma \mathbf{X}_i^{'}\mathbf{b}} (\lambda_1 t_{1i}^{\sigma} (\log t_{1i} - \mathbf{X}_i^{'}\mathbf{b}) + \lambda_2 T_{2i}^{\sigma} (\log t_{2i} - \mathbf{X}_i^{'}\mathbf{b}) + \\ & \lambda_3 t_{(2)i}^{\sigma} (\log t_{(2)i} - \mathbf{X}_i^{'}\mathbf{b})) \qquad j = 1,...,m\end{aligned} \tag{52}$$

The observed Fisher information matrix, $\mathbf{I}$ is a $(m+4) \times (m+4)$ matrix, where the entries are second order partial derivatives displayed above.

$$\mathbf{I} = -\begin{pmatrix} \frac{\partial^2 logL}{\partial\lambda_i\partial\lambda_j} & \frac{\partial^2 logL}{\partial\lambda_i\partial\sigma} & \frac{\partial^2 logL}{\partial\lambda_i\partial b_l} \\ \frac{\partial^2 logL}{\partial\lambda_j\partial\sigma} & \frac{\partial^2 logL}{\partial\sigma^2} & \frac{\partial^2 logL}{\partial\sigma\partial b_l} \\ \frac{\partial^2 logL}{\partial\lambda_j\partial b_k} & \frac{\partial^2 logL}{\partial\sigma\partial b_k} & \frac{\partial^2 logL}{\partial b_l\partial b_k} \end{pmatrix}.$$

The inverse of the observed Fisher information matrix is the observed variance-covariance matrix

$$\mathbf{\Sigma} = \mathbf{I}^{-1} \tag{53}$$

of the ML-estimator $\underline{\hat{\theta}} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\sigma}, \hat{b}_1, ..., \hat{b}_m)'$ of the regression parameter $\underline{\theta} = (\lambda_1, \lambda_2, \lambda_3, \sigma, b_1, ..., b_m)'$.

The quantity $\sqrt{n}(\underline{\hat{\theta}} - \underline{\theta})$ has an asymptotic multivariate normal distribution with mean vector zero and observed variance-covariance matrix $\mathbf{\Sigma}$.

## 4 Significance Test for the Regression Coefficients and Fitting the Model

The hypotheses about $\mathbf{b}$ can be frequently put in the form $H_o : \mathbf{b}_1 = 0$, with $\mathbf{b}$ partitioned as $\mathbf{b}^{'} = (\mathbf{b}_1, \mathbf{b}_2)^{'}$ where $\mathbf{b}_1$ is a $k$-dimensional vector with $(k < m)$. To test $H_0$ against the alternative that $\mathbf{b}_1 \neq 0$ one can use

$$\Lambda_1 = \hat{\mathbf{b}}_1^{'} \Sigma_{11}^{-1} \hat{\mathbf{b}}_1 \qquad (5)$$

where $\Sigma_{11}$ is $k \times k$ asymptotic observed variance-covariance matrix of $\hat{\mathbf{b}}_1$. Under $H_0$, $\Lambda_1$ is asymptotically chi-square with $k$ degrees of freedom.

For fitting the regression model, we use forward selection method. The procedure is as follows. We take one covariate at a time in the regression model and testing for the corresponding regression coefficient. Go on adding the covariates in the regression model until all regression coefficients in the model are significant at least at 5% level of significance.

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David D. Hanagal[1]
Department of Statistics in Medicine
Heinrich Heine University
Düsseldorf, Germany

[1]On leave from Department of Statistics, University of Pune, Pune-411007, India.